FIGURE 2.12   Resistivity data taken with a high purity aluminum rod as the sample. The decay is clearly not described by a single exponential at the earlier times.

should be clear. You may see some transient oscillations of the signal right after the field shuts off, but there should be plenty of time left after these oscillations die away for you to get a smooth curve. Figure 2.12 shows data acquired with a $\frac{1}{2}$-in. diameter high-purity aluminum rod[10] at room temperature as a sample. The data points are the output of a digital oscilloscope displayed using MATLAB. Note that at the earliest times, there are higher order contributions to the signal (as described by Bean *et al.*), and one must choose a suitable range over which the data are indeed described by a single exponential.

The fit shown in Fig. 2.12 yields a decay time $t_E = 3.051 \times 10^{-4}$ s. Then, from Eq. (2.16) we find for the resistivity

$$\rho = \frac{2.17 \times 10^{-9}}{t_E \text{ (s)}} \times r^2 \text{ (cm}^2) = 2.87 \times 10^{-6} \ \Omega \cdot \text{cm},$$

where we used the fitted value of t_E and $r = 0.635$ cm. This compares well with the value listed in Table 2.1.

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[10]From the Alfa Aesar company, http://www.alfa.com/.

The main source of systematic uncertainty is likely to come from the times over which the decaying voltage signal is fitted. At short times, the decay is not a pure exponential because the transient terms have not all died away, so we want to exclude these times when we fit. At long times, there may be some left over voltage level that is a constant added to the exponential, and again, a pure exponential fit will be wrong. Varying the upper and lower fit limits until we get a set that gives the same answer as a set that is a little bit larger on both ends is one approach. One should be convinced that the results are consistent. For example, use aluminum alloy rods of the same composition but different radii, and check to make sure that the decay lifetimes t_E scale like r^2. This should certainly be the case to within the estimated experimental uncertainty.

Having learned how to take and analyze data on resistivity, we can now investigate the temperature dependence. It is best to start simply by comparing the two samples of $\frac{1}{2}$-in. diameter aluminum rods, one an alloy and the other a (relatively) pure metal. Vary the temperature by immersing the samples in baths of ice water, dry ice and alcohol, and liquid nitrogen. Boiling water or hot oil can also be used. These measurements are tricky. One must remove the sample from the bath and measure the eddy current decay before the temperature changes very much. Probably the best way to do this is to take a single trace right after inserting the sample, stop the oscilloscope, and store the trace. Then one analyzes the trace offline to get the decay constant. One might also try to estimate how fast the bar warms up by making additional measurements after waiting several seconds, e.g., after saving the trace. This would best be done with a sample whose resistivity, and therefore t_E, can be expected to change a lot with temperature. Pure aluminum is a good choice. Remember that the temperature dependence will be much different for the pure metal than for the alloy. Try to estimate the contribution to the mean free path of the electrons due to the impurities.

2.3.   EXPERIMENT ON THE HALL EFFECT

In Section 2.2 we saw how collisions of electrons with the crystal lattice lead to an electrical resistance, when those electrons are forced to move under an electric field. If one also applies a magnetic field, in a direction perpendicular to the electric field, then the electrons (and other current carriers) will be deflected sideways. As a result an electric field appears in this direction, and therefore also a potential difference. This phenomenon

is called the Hall effect, and has important applications both in identifying the current carriers in a material and for practical use as a technique for measuring magnetic fields.

Let us rewrite the microscopic formula for Ohm's law, but this time taking care to indicate current density and electric fields as vectors, and to also note the negative sign of the charge on the electron. Following Eqs. (2.12) and (2.13) we write

$$\mathbf{j} = -ne\mathbf{v_d} = ne^2\tau \mathbf{E}/m \qquad (2.18)$$

or

$$\frac{m\mathbf{v_d}}{\tau} = -e\mathbf{E}. \qquad (2.19)$$

It is clear that in Eq. (2.19) we have made an approximation, replacing the time rate of change of momentum, i.e., $d\mathbf{p}/dt = md\mathbf{v}/dt$, with an expression that uses the average acceleration $\mathbf{v_d}/\tau$. This is how we have taken into account collisions with the crystal lattice.

It is straightforward to modify Eq. (2.19) to take into account the effect of a magnetic field $\mathbf{B}$. We have

$$\frac{m\mathbf{v_d}}{\tau} = -e(\mathbf{E} + \mathbf{v_d} \times \mathbf{B}).$$

If we assume that the magnetic field lies in the z direction, and define the cyclotron frequency $\omega_c \equiv eB/m$, then we can rewrite this equation as

$$v_{d_x} = -\frac{e\tau}{m}E_x - \omega_c\tau v_{d_y}$$

$$v_{d_y} = -\frac{e\tau}{m}E_y + \omega_c\tau v_{d_x} \qquad (2.20)$$

$$v_{d_z} = -\frac{e\tau}{m}E_z.$$

Consider now a long rectangular section of a conductor, as shown in Fig. 2.13. A longitudinal electric field E_x is applied, leading to a current density flowing in the x direction. As this electric field is initially turned on, the magnetic field deflects electrons along the y direction. This leads to a buildup of charge on the faces parallel to the xz plane, and therefore an electric field E_y within the conductor. In the steady state, this electric field cancels the force due to the magnetic field, and the current density is strictly
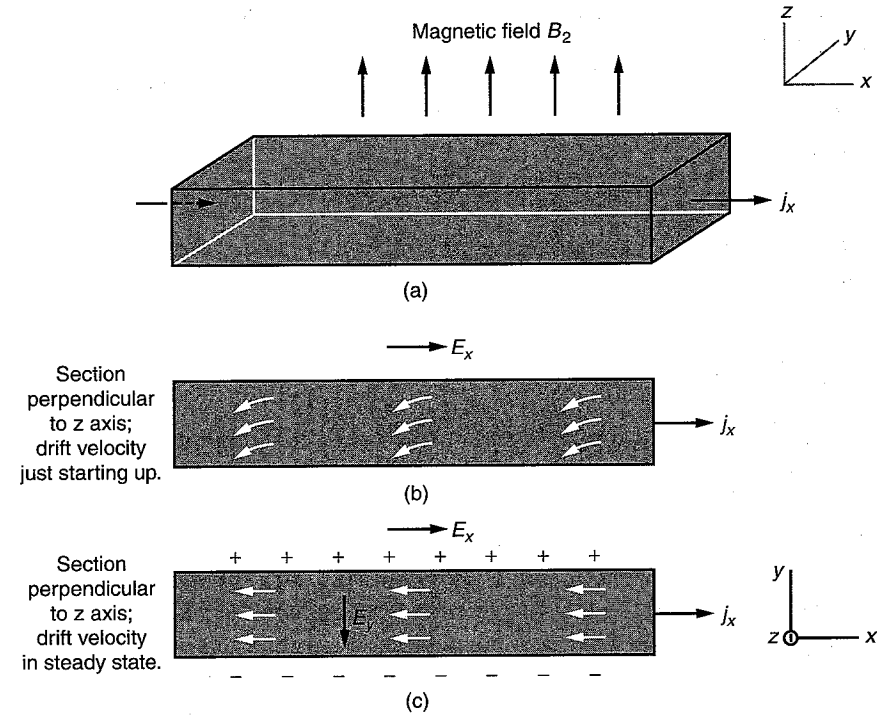

FIGURE 2.13   The standard geometry for discussing the Hall effect (after Kittel).

in the x direction, hence $v_{d_y} = 0$. From Eqs. (2.20) we therefore have

$$E_y = \frac{m\omega_c}{e}v_{d_x} = \frac{m\omega_c}{e}\left(-\frac{e\tau}{m}E_x\right) = -\omega_c\tau E_x = -\frac{eB\tau}{m}E_x.$$

The appearance of the electric field E_y is the Hall effect.

A convenient experimental quantity is the Hall coefficient R_H, defined as

$$R_H \equiv \frac{E_y}{j_x B} \qquad (2.21)$$

The quantities E_y, j_x, and B are all straightforward to measure, and in our simple approximation for electrons in conductors we have (from Eq. (2.18)) $j_x = ne^2\tau E_x/m$; therefore,

$$R_H = \frac{eB\tau E_x/m}{(ne^2\tau E_x/m)B} = \frac{1}{ne}. \qquad (2.22)$$

That is, the Hall coefficient is the inverse of the carrier charge density. In fact, the Hall effect is a useful way to measure the concentration of charge carriers in a conductor. It is also convenient to define the Hall resistivity as the ratio of the transverse electric field to the longitudinal current density, that is,

$$\rho_{\rm H} \equiv E_y/j_x = B R_{\rm H}, \qquad (2.23)$$

which depends (in our approximation) only on the material and the applied magnetic field.

2.3.1. Measurements

In order to measure the Hall effect, one needs a sample of a conductor, but not an especially good conductor. This is because one also needs a relatively low carrier density ne in order to get a sizable effect; this of course leads to a relatively high resistivity. As seen in Table 2.1, bismuth is a good candidate metal, and we describe such an experiment here.[11]

The setup uses a bismuth sample with rectangular cross section, mounted on a probe with attached leads for measuring current and voltage. A thermocouple is also attached to the sample so that temperature measurements can be carried out. The magnetic field is provided by an electromagnet capable of delivering a field up to $\sim$5 kG over a volume roughly 1 cm^3. The bismuth sample probe is shown in Fig. 2.14. The width of the bismuth sample is $w = 6.5$ mm and its thickness, measured with a micrometer, is $t = 1.65 \times 10^{-4}$ m. The effective length of the sample is the distance between the leads used to measure the current ("white" and "brown," as shown in Fig. 2.14). In our case, this distance is $\ell = 7$ mm. Current is supplied by a DC power supply, connected to the sample through the "red" and "black" leads. The Hall voltage is measured with a digital multimeter, using the "green" lead and the output of a potentiometer used to balance the voltage on the "white" and "brown" leads. A separate bundle of wires are connected to leads that carry current to the heating resistor, and to a thermocouple that measures the temperature of the bismuth sample.

Begin by determining the Hall coefficient at room temperature and for a relatively high magnetic field. Turn on the electromagnet power supply to
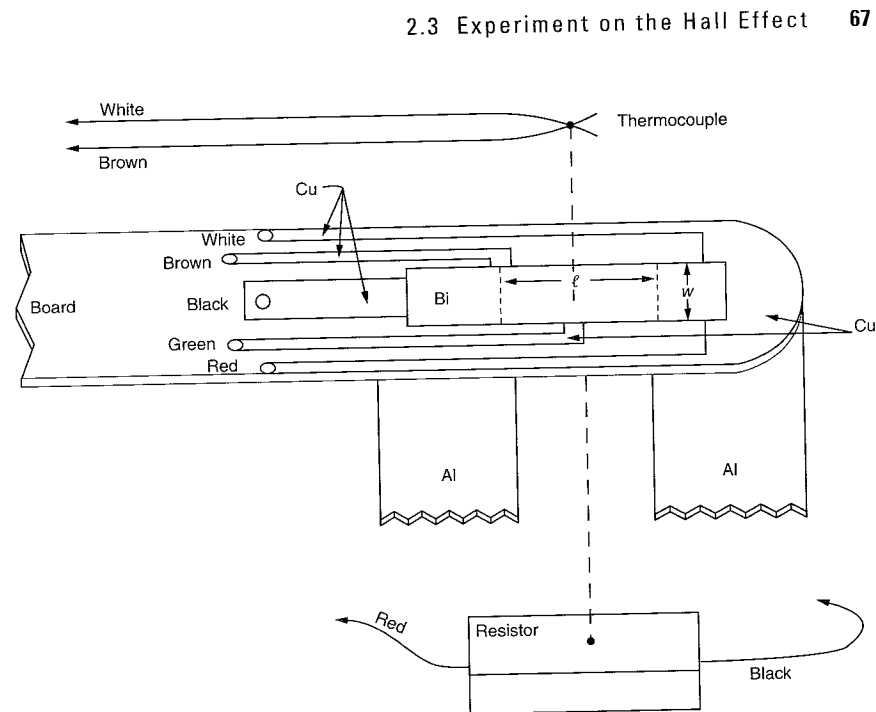

FIGURE 2.14 Schematic of the probe used to make measurements of the Hall effect in bismuth. Electrical connections are made to the bismuth sample using copper leads. A thermocouple, as well as a resistor which acts as a heat source, is also attached to the sample. Two separate bundles of wires emerge from the probe, one of which is used exclusively for heating the sample and for measuring its temperature.

around 4 kG. It will likely need an hour or so to stabilize. In the meantime, with the sample probe removed from the magnetic field, run about 3 A through the bismuth sample, and adjust the potentiometer so that the Hall voltage is zero. Return the current through the sample to zero. *The sample can get quite hot while it is conducting so much current. Be careful not to touch it, or to touch it to anything else.*

When the electromagnet is stabilized, measure and record the magnetic field using a gaussmeter, or by some other technique. Now, place the sample probe in the center of the magnetic field. Quickly raise the current I through the sample to 3.0 A, and record the Hall voltage $V_{\rm H}$. Then, quickly, reduce the current by 0.25 A, and record the Hall voltage again. You should carry this series of measurements out rather rapidly to avoid leaving the bismuth sample at high temperature for any extended period of time. When you have reduced the current to near zero, and recorded the final value of the

[11] Semiconductors also make good candidates, with a very low carrier density compared to a metal. For a description of such a setup, see A. Melissinos, *Experiments in Modern Physics*, First ed., Academic Press, New York, 1966.
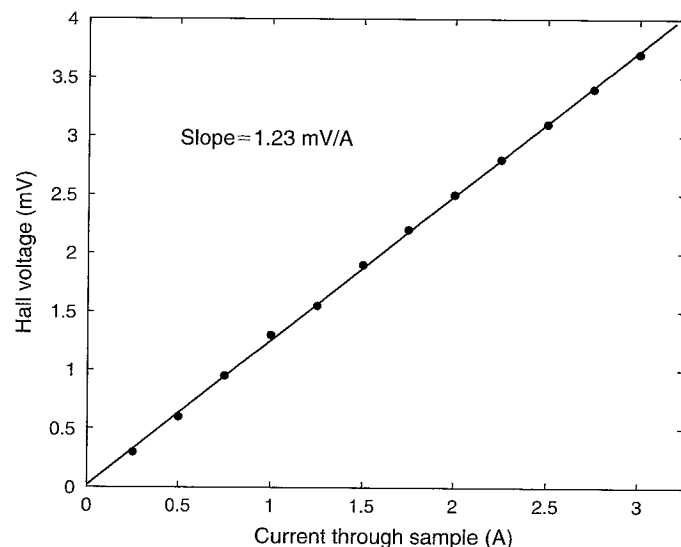
FIGURE 2.15 Sample of Hall effect data, taken at room temperature and with a magnetic field $B = 4.42$ kG.

Hall voltage, remove the probe and recheck the value of the magnetic field.

A sample of data taken in this way, at room temperature and with $B = 4.42$ kG, is shown in Fig. 2.15. A free linear straight line fit gives a slope of 1.23 mV/A, with an intercept very close to zero. In terms of quantities related to our measurement, the Hall coefficient (Eq. (2.21)) is expressed by

$$R_H \equiv \frac{E_y}{j_x B} = \frac{V_H/w}{I/(w \times t)B} = \frac{V_H t}{I B} = \frac{d V_H}{d I}\frac{t}{B},$$

where we note that our data yields a very good direct proportional relationship between V_H and I. Using SI units, this yields

$$R_H = \left(1.23 \times 10^{-3}\frac{V}{A}\right)\left(\frac{1.65 \times 10^{-4}\ m}{0.442\ T}\right) = 4.59 \times 10^{-7}\ m^3/C$$

This is quite close to an accepted room temperature value of $R_H = 5.4 \times 10^{-7}\ m^3/C$ for pure bismuth metal. The uncertainties in measuring the dimensions of the sample can easily account for the discrepancy.

Of course, this sample and this setup can be used to determine the resistivity of bismuth. Outside of the magnetic field, measure the voltage

TABLE 2.2 Sample data, taken by a student, for the resistivity ρ of bismuth as a function of temperature, using the Hall effect apparatus

T (°C)	T (K)	ρ ($\mu\Omega$-cm)
−80	193	70
−60	213	85
−40	233	96
−20	253	110
0	273	121
20	293	134
40	313	150
60	333	163

drop along the length ℓ of the bismuth sample, as a function of the applied current, and determine the resistivity ρ from the ratio

$$\rho = \frac{E_x}{j_x} = \frac{d V_x}{d I}\frac{wt}{\ell}.$$

The temperature dependence of each of these quantities can be determined by heating (and cooling) the probe, and recording values as a function of temperature using readings from the thermocouple.

Table 2.2 lists some results for the resistivity ρ in ($\mu\Omega$-cm) as a function of temperature. To examine the temperature dependence it is best to make a log–log plot of the data vs T since we expect a power law dependence. This is shown in Fig. 2.16 and when fitted gives

$$\rho \propto T^{1.52}.$$

Note that at room temperature ($T = 25$°C)

$$\rho = 1.4 \times 10^{-4}\ \Omega\text{-cm}$$

in reasonable agreement with the data of Table 2.1.

Indeed, one expects a $T^{3/2}$ dependence of the resistivity on the temperature because of the following argument. From Eq. (2.14) the resistivity is inversely proportional to the mean time between collisions, as long as the carrier density remains constant. Now the mean time between collisions is given by
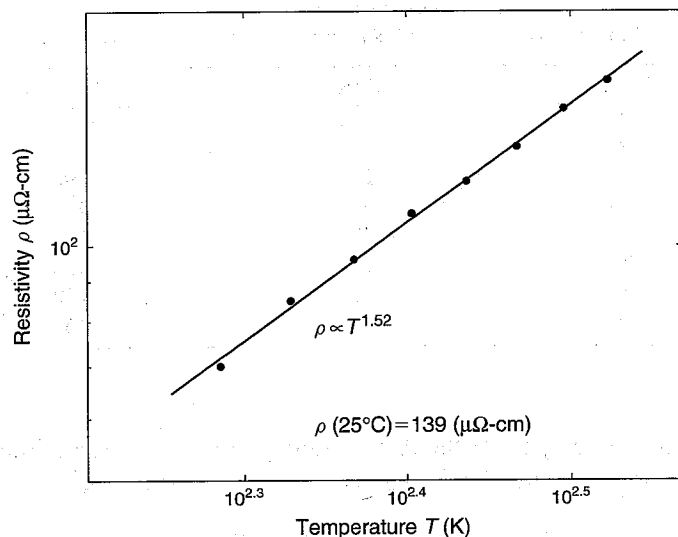
$$\tau = \lambda/v,$$

FIGURE 2.16   The resistivity of bismuth as a function of temperature, taken with the Hall effect apparatus (data from Table 2.2.) The data are fitted to a power law form.

where λ is the mean free path for scattering, and v the thermal velocity of the electrons. For v we can use

$$\frac{1}{2}mv^2 = \frac{3}{2}kT \quad \text{or} \quad v = \sqrt{3kT/m}.$$

The mean free path, λ, decreases as the collision cross section increases, namely as the lattice vibrations increase with temperature. It is found that λ is inversely proportional to the temperature, and therefore

$$\tau \propto 1/T^{3/2}$$

or using Eq. (2.14),

$$\rho \propto T^{3/2}.$$

We can also examine the temperature dependence of the Hall coefficient. In this case it is best to plot R_H on a semi-log plot vs $1/T$. The reason is that the Hall coefficient (see Eq. (2.22)) is directly inversely proportional to the carrier density, and we expect the carrier density to depend on the temperature by an exponential factor, such as for instance shown in Eq. (2.28). The data are plotted in this way in Fig. 2.17, and we recognize two distinct slopes. As expected, R_H falls with increasing temperature
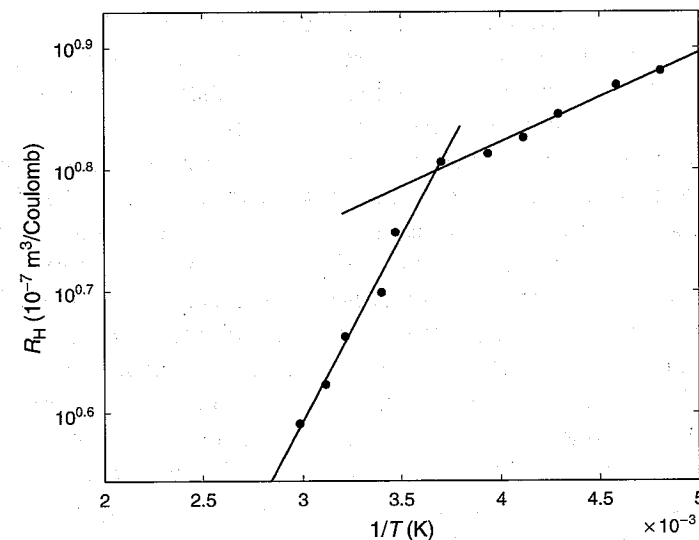
FIGURE 2.17   Measurements of the Hall coefficient as a function of temperature.

because the carrier density increases. By fitting the data to the form

$$n \propto \exp(-E/2kT),$$

we find for the two regions

low T, $E = 0.029$ eV
high T, $E = 0.120$ eV.

Such energy differences are typical of the excitation of impurities. It is also relevant to note that the carrier density at room temperature is

$$n = 1/eR_H = 1.35 \times 10^{19}\,\text{cm}^{-3}.$$

This is quite high and typical of a conductor.

2.4. SEMICONDUCTORS

2.4.1. General Properties of Semiconductors

We have seen in the first section how a free-electron gas behaves, and what can be expected for the band structure of a crystalline solid. In the second

section we applied the model of a free-electron gas to the behavior of the resistivity of metals. In the present section we will study some properties of semiconductors that can be verified easily in the laboratory, where we will make use both of the free electron gas model and of the band structure of the material. As mentioned before, a semiconductor is a crystalline solid in which the conduction band lies close to the valence band, but is not populated at low temperatures; semiconductors are unlike most metals in that both *electrons* and *holes* are responsible for the properties of the semiconductor. If the semiconductor is a pure crystal, the number of holes (positive carriers, p) is equal to the number of free electrons (negative carriers, n), since for each electron raised to the conduction band, a hole is created in the valence band: these are called the *intrinsic* carriers. All practically important semiconductor materials, however, have in them a certain amount of impurities that are capable either of donating electrons to the conduction band (making an n-type crystal) or of accepting electrons from the valence band, thus creating holes in it (making a p-type crystal). These impurities are called *extrinsic* carriers and in such crystals $n \neq p$.

Let us then first look at the energy-band picture of a semiconductor as it is shown in Fig. 2.18; the impurities are all concentrated at a single energy level usually lying close to, but below, the conduction band. The density of states must be different from that of a free-electron gas (Eq. (2.4) and Fig. 2.2a) since, for example, in the forbidden gaps it must be 0; close to the ends of the allowed bands it varies as $E^{1/2}$ and reduces to 0 on the edge.
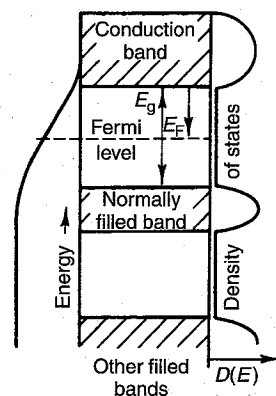

FIGURE 2.18   Energy band structure of a semiconductor without impurities. On the left-hand side the Fermi distribution for a free-electron gas is shown; on the right-hand side the actual density of states $D(E)$ is shown.

On the other hand, the Fermi distribution function, Eq. (2.3), remains the same. The only parameter in this function is the Fermi energy, which can be found by integrating the number of *occupied* states (Fermi function times density of states) and setting it equal to the electron density. It is clear, however, that if we are to have as many empty states in the valence band as occupied ones in the conduction band, the Fermi level must lie exactly in the middle of the forbidden gap[12] (because of the symmetry of the trailing edge of the distribution). In Fig. 2.18, the density of states is shown to the right and the Fermi distribution function to the left. We measure the position of the Fermi level *from the conduction band* and define it by E_F; the exact value of E_F is

$$E_F = -\frac{E_g}{2} + kT \ln \left(\frac{m_h^*}{m_e^*} \right)^{3/4}. \qquad (2.24)$$

Since the Fermi level lies below the conduction band, E_F is a *negative* quantity, E_g is the energy gap always taken to be positive, and m_h^* and m_e^* are the effective masses of holes and electrons, respectively. If w_C and w_F stand for the actual position of the conduction band and Fermi level above the zero point energy, then

$$w_F = w_C + E_F.$$

To find the density of electrons in the conduction band (or holes in the valence band) we simply substitute Eq. (2.24) for w_F into Eq. (2.4), multiply by the density of states, and integrate over w from $w = w_C$ to $+\infty$. When, however, the exponent

$$-(w_F - w) \approx \frac{E_g}{2} + E \gg kT, \qquad (2.25)$$

the Fermi distribution degenerates to a Boltzmann distribution. (Here E is the energy of the electrons as measured from the top of the conduction band; obviously it can take either positive or negative values.) With this *assumption* the integration is easy, yielding

$$n = \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{E_F/kT} \approx \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-E_g/2kT}; \qquad (2.26)$$

[12]If the effective masses of p- and n-type carriers are the same.

similarly,

$$p = \left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{-(E_g + E_F)/kT} \approx \left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{-E_g/2kT}. \quad (2.27)$$

It is interesting that the product np is independent of the position of the Fermi level[13]—especially if we take $m_e = m_h$

$$n_i^2 = np = 2.31 \times 10^{31} T^3 e^{-E_g/kT}.$$

From the analysis we expect that as the temperature is raised, the density of the intrinsic carriers in a semiconductor will increase at an exponential rate characterized by $E_g/2kT$. This temperature is usually very high since $E_g \approx 0.7$ V (see Eqs. (2.29)).

We have already mentioned that impurities determine the properties of a semiconductor, especially at low temperatures where very few intrinsic carriers are populating the conduction band. These impurities, when in their ground state, are usually concentrated in a single energy level lying very close to the conduction band (if they are donor impurities) or very close to the valence band (if they are acceptors). As for the intrinsic carriers, the Fermi level for the impurity carries lies halfway between the conduction (valence) band and the impurity level; this situation is shown in Figs. 2.19a and 2.19b. If we make again the low temperature approximation of Eq. (2.25), the electron density in the conduction band is given by

$$n = N_d \left(\frac{2\pi m kT}{h^2}\right)^{3/2} e^{-E_d/2kT}, \quad (2.28)$$

where N_d is the donor density and E_d the separation of the donor energy level from the conduction band. In writing Eq. (2.28), however, care must be exercised because the conditions of Eq. (2.25) are valid only for very low temperatures. Note, for example, that for germanium

$$E_g = 0.7 \text{ eV}, \quad \text{and for } kT = 0.7 \text{ eV}, \quad T = 8000 \text{ K}$$

whereas

$$E_d = 0.01 \text{ eV}, \quad \text{and for } kT = 0.01 \text{ eV}, \quad T = 120 \text{ K}. \quad (2.29)$$

Thus at temperatures $T \approx 120$ K most of the donor impurities will be in the conduction band and instead of Eq. (2.28) we will have $n \approx N_d$; namely,

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[13]This result is very general and holds even without the approximation that led to Eqs. (2.26) and (2.27).
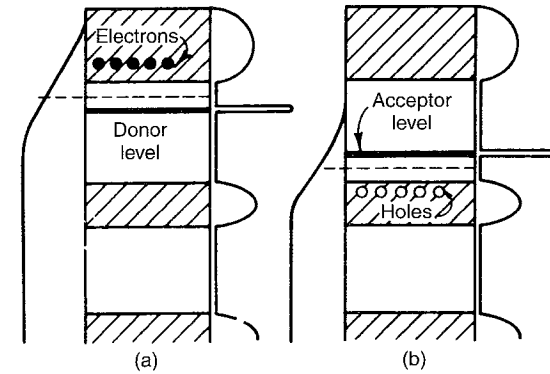

FIGURE 2.19   Same as described in the legend to Fig. 2.18 but with the addition of impurities. (a) The impurities are of the donor type and lie at an energy slightly below the conduction band. (b) The impurities are of the acceptor type and lie slightly above the valence band. Note the shift of the Fermi level as indicated by the dotted line.

the density of impurity carriers becomes saturated. Once saturation has been reached the impurity carriers in the conduction band behave like the free electrons of a metal.

2.4.2. Sketch of p–n Semiconductor Junction Theory

Semiconductor materials with high impurity concentration, when properly combined, form a transistor. Junction transistors consist of two junctions of dissimilar-type semiconductors, one p type and one n type; the intermediate region, the base, is usually made very thin. We will briefly sketch the behavior of such a p–n junction and then see how the combination of two junctions can provide power amplification; for this we will use our knowledge of the band structure of semiconductors and the position of the Fermi level, as developed previously (Figs. 2.18 and 2.19). When two materials with dissimilar band structure are joined, it is important to know at what relative energy level one band diagram lies with respect to the other: the answer is that *the Fermi levels of both materials must be at the same energy position* when no external fields are applied; this is shown in Fig. 2.20.

From the energy diagram of Fig. 2.20, it follows that only electrons with $E_e > \Delta W_e$ will be able to cross the junction from the n material into the p region and only holes with $E_h > \Delta W_h$ from the p region into the n region.