Thermal Johnson Noise Generated by a Resistor

Complete Pre-Lab before starting this experiment

HISTORY

In 1926, experimental physicist John Johnson working in the physics division at Bell Labs was researching noise in electronic circuits. He discovered that there was an irreducible low level of noise in resistors whose power was proportional to temperature. Harry Nyquist, a theorist in that division, got interested in the phenomenon and developed an elegant explanation based on fundamental physics.

THEORY OF THERMAL JOHNSON NOISE

Thermal agitation of electrons in a resistor gives rise to random fluctuations in the voltage across its terminals, known as Johnson noise. In Problem 1, you are to show that in a narrow band of frequencies, Δf, the contribution to the mean-squared noise voltage from this thermal agitation is,

$$\left\langle V(t)^2 \right\rangle_{time} = 4 R k_B T \Delta f \tag{1}$$

where R is the resistance in ohms and T is the temperature in degrees Kelvin for the resistor, k_B is the Boltzmann constant (1.38×10^{-23} J/K).

This voltage is too small to be detected without amplification. If the resistor is connected across the input of a high-gain amplifier whose voltage gain as a function of frequency is $G(f)$, the mean square of the voltage output of the amplifier will be:

$$\left\langle V_{out}(t)^2 \right\rangle_{time} = 4 R k_B T \int_0^\infty [G(f)]^2 df + \left\langle V(t)_N^2 \right\rangle_{time} \tag{2}$$

where $\left\langle V(t)_N^2 \right\rangle_{time}$ is the output noise generated by the amplifier itself.

Thus by measuring $\left\langle V_{out}(t)^2 \right\rangle_{time}$ as a function of R and making a plot, one obtains $4 k_B T \int_0^\infty [G(f)]^2 df$ from the slope, while the abscissa gives $\left\langle V(t)_N^2 \right\rangle_{time}$. But the amplifier gain

$G(f)$ can be independently measured and the gain integral $\int_0^\infty [G(f)]^2 df$ evaluated. The slope will then give a value for the Boltzmann constant k_B.

This is in outline the first part of the experiment. The second part involves measuring the noise voltage as a function of the temperature, to verify the expected temperature dependence.

Problem 1 - Derivation of Eq. (1)

An electrical transmission line connected at one end to a resistor R and at the other end by an "equivalent" resistor R may be treated as a one-dimensional example of black body radiation.
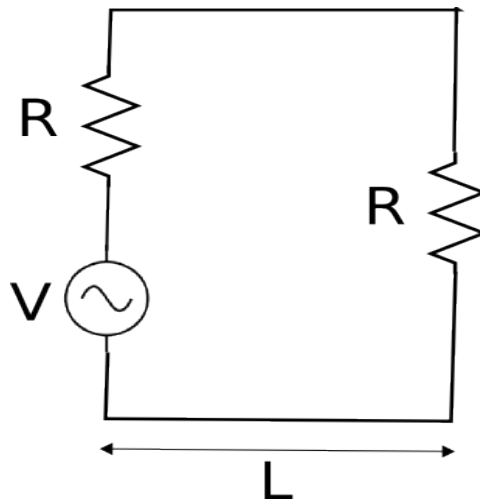
Figure 1. Two resistors (of equal resistance R) coupled via a transmission line.

At finite temperature T, the resistor R generates a noise voltage $V(t)$ which will propagate down the line. If the characteristic impedance of the transmission line is made equal to R, the radiation incident on the "equivalent" resistor R from the first resistor R should be completely absorbed.

The permitted standing wave modes in the line have $\lambda = 2L/n$ and $f = (c/2L)n$, where $n = 1, 2, 3$, etc., and v is the wave velocity in the line. The separation of the modes in frequency is $v/2L$, mode density σ, and the number of modes between f and $f + \Delta f$ is

$$\sigma(f)\Delta f = (2L/c)\Delta f \tag{3}$$

From the Planck distribution or the equipartition theorem, the mean thermal energy contained in each electromagnetic mode or photon state in the transmission line is:

$$\langle E(f) \rangle = \frac{hf}{e^{hf/k_B T} - 1} \approx k_B T \quad \text{at low frequencies.} \tag{4}$$

From Eq. (3) and (4) find the electromagnetic energy $\langle E(f)\rangle \sigma(f)\Delta f$ in a frequency interval Δf. One half of this energy is generated by the first resistor of R and propagating towards the "equivalent" resistor R. Knowing the propagation time from the generating resistor to the absorbing resistor $\Delta t = L/c$, show that the absorbed power by the "equivalent" resistor R equals

$$P(f)\Delta f = k_B T \Delta f. \tag{5}$$

In thermal equilibrium, this power is simply the ohmic heating generated by a noise voltage source $V(t)$ from the first resistor. Since $V(t)$ is terminated by the absorbing resistor R and has an "internal" resistance R (the first resistor), it produces a current $I = V/(2R)$ in the line. Hence the power absorbed by the "equivalent" resistor R over the frequency interval Δf can also be calculated as

$$I^2 R = \left(\frac{V}{2R}\right)^2 R = \frac{V^2}{4R} = \frac{V^2(f)\Delta f}{4R} \tag{6}$$

By equating $P(f)\Delta f = k_B T \Delta f$ to $V^2 \Delta f / R$, show that

$$V^2(f)\Delta f = 4k_B TR\Delta f \tag{7}$$

and

$$\langle V(t)^2 \rangle_{time} = 4k_B TR\Delta f \tag{8}$$

This is known as Nyquist's theorem as shown in Eq. (1). The power spectral density (noise power per unit frequency) is independent of frequency. Most other noise sources in nature have a f^{-1} to f^{-2} spectrum.

   Question: what is the integrated power of this Johnson noise over all frequencies? [i.e., why can't a single resistor supply the world's energy needs?]

   Eq. (1) is interesting: the left hand side describes random fluctuations of a lossy system in thermodynamic equilibrium. i.e. it is an equilibrium property; both the electrons and the lattice are in equilibrium at some temperature T. However, the right hand side of the equation refers explicitly to a non-equilibrium property of the same lossy system: the resistance R, which is measured by applying a voltage (taking the system out of equilibrium) and measuring the current of electrons scattering through the lossy system.
Johnson noise is an example of a broader fundamental principle in nature called the "*fluctuation-dissipation theorem*." It relates the non-equilibrium dissipation in a system to its spontaneous fluctuations in equilibrium. See accompanying papers by Callen.


RESISTOR AND APPARATUS

   You have several resistors for your source of noise: a box with many resistors at room temperature

(switch selected), and a resistor connected at the end of parallel wire leads in a stainless steel tube. The resistor is thermally connected to the brass bottom plate of the tube with some glue. The wire leads are thin so they don't conduct much heat; this is so that the resistor at the end of the tube will cool to the temperature of the bottom plate when cooled.

A very low noise operational amplifier is used as the first stage of amplification for the Johnson noise. This preamp has a built-in band pass filter: ~100 Hz to 1800 Hz. Beware: the filter built into the amplifier is only one component in the effective gain vs frequency behavior for this system: the small "parasitic" capacitance in parallel with the resistor is a filter in its own right (you must account for this). The resulting "filter" is a product of the RC filter on the input (small high frequency roll-off) and the rather square bandpass filter in the amplifier box. The apparatus is connected with shielded coaxial cables as shown to reduce pickup. *Beware: there is a special easily damaged 3-wire connector on the amplifier input that looks like a BNC, but it is not! Look at it carefully. Ask us to help you with it.*

The sine wave oscillator is used to measure the gain vs frequency of the amplifier. The oscillator output is put through an attenuator to reduce it to the level needed to be able to insert into the amplifier. The attenuation factor is accurately given by the controls and does not need to be calibrated. Check its output voltage before directly connecting to the amplifier input. The frequency f of the oscillator can be accurately set and determined with the Integrating Digital Voltmeter. Finally, you have a spectrum analyzer.
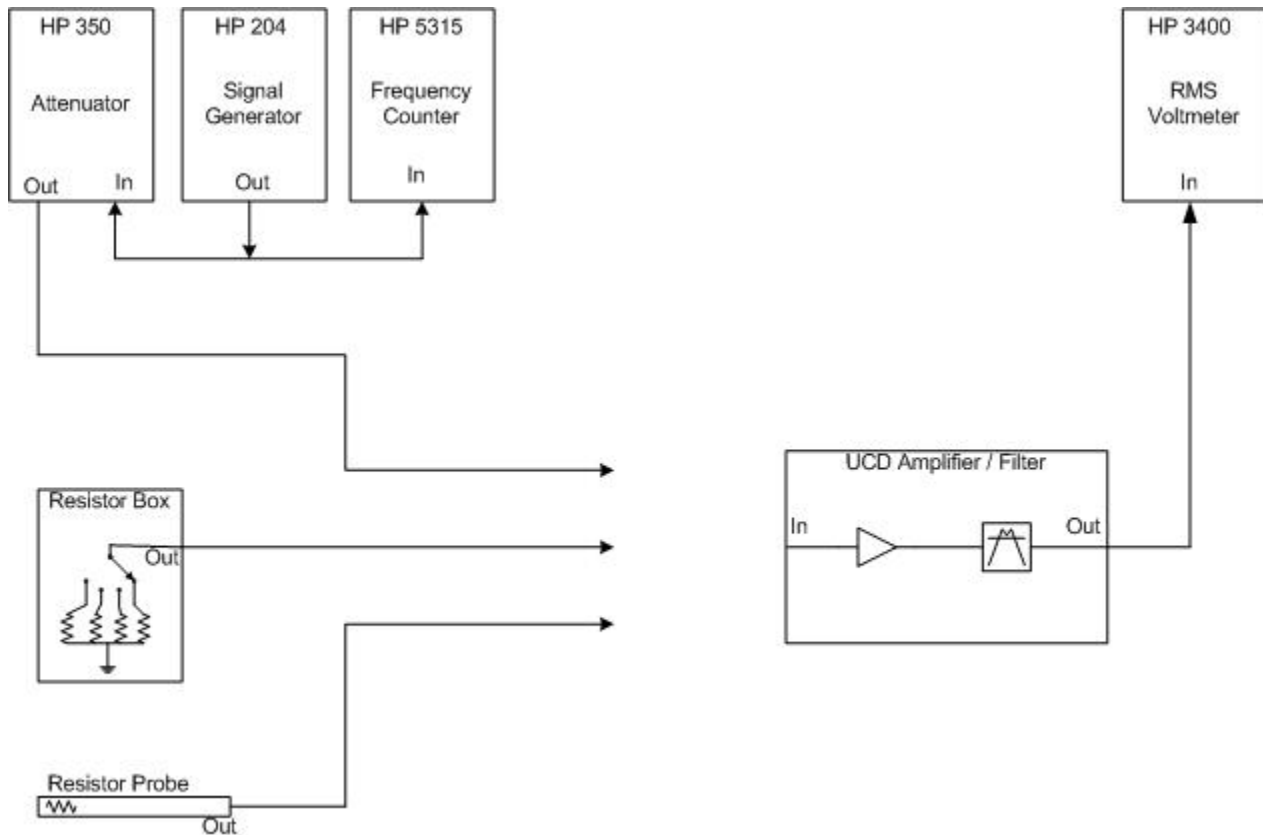
In addition to the resistors (box and cold probe) and the low noise bandpass amplifier shown above, you have all the equipment you need (and more!) to design your experiment and make the necessary measurements, including calibration. *Shown above (left to right):*

HP 5315 Frequency counter
Agilent 34401A Digital precision volt meter (AC or DC)
HP 204D Oscillator
HP350D Attenuator
Power supply for the low noise band pass amplifier
Tektronix 2205 Oscilloscope
Computer with software for PicoScope and data analysis software
PicoScope 4262 Digital scope and spectrum analyzer (off photo to right)
See http://www.picotech.com/applications/resolution.html
SR770 Spectrum Analyzer

We recommend that you first get a feel for the measurement by making a rough approximation. Look at the spectrum on the PicoScope [instructions below] and make an approximation for an equivalent square bandpass. Use that Δf as the equivalent bandpass in the equation for Johnson noise.   To get the gain of the amplifier, inject a small amplitude sine wave as a calibrator as

described in the next section and measure the output. Use the RMS AC voltmeter in all these measurements. With some care you should be able to get the Boltzmann constant to 20% accuracy in this way. A more accurate measurement involves integrating over the squared gain [since you are measuring squared voltage, or power] as described in the next section.

PROCEDURE FOR MEASURING THE GAIN INTEGRAL $\int_0^\infty [G(f)]^2 df$

You obtain $\int_0^\infty [G(f)]^2 df$ by measuring the amplifier gain $G(f)$ at a discrete and evenly spaced set of frequency values (f) and then evaluating the discrete sum $\sum [G(f)]^2 \Delta f$ numerically. You measure the gain G by injecting a known voltage to the amplifier and then measuring its output voltage. But: the amplifier takes only VERY small input voltages!! **Keep input under 250 microV**.

To measure $G(f)$, connect a precision broadband voltage attenuator to the input of the amplifier. The attenuator is used to assist you in determining the amplifier gain $G(f)$ as follows. Supply the input of the attenuator with a sinusoidal voltage signal of $V_i = 1$ volt at some frequency between zero and 3000 Hz. For the best accuracy, measure the applied voltage signal with a digital rms voltmeter. Since at f = 1000 Hz, $G(f)$ is roughly 10000, you can set the attenuation parameter $N_{dB} \sim 80\,dB$ that gives a voltage attenuation factor of $G_A = 10^{N_{dB}/20} \sim 10000$. The output of the attenuator, V_i/G_A, is then fed into the input of the amplifier. You measure the amplifier output $V_o = (V_i/G_A)G(f)$ with the same rms voltmeter. The amplifier gain is given by

$$G(f) = (V_o/V_i)G_A = (V_o/V_i)10^{N_{dB}/20} \qquad (9)$$

It is best that you adjust N_{dB} so that $V_o/V_i \sim 1$. First get this working at 1000 Hz. Repeat the measurement at a series of frequencies, to obtain a discrete set of $G(f)$ values. From these results, you can numerically calculate $\int_0^\infty [G(f)]^2 df$.

There is an easier way: once you have a reliable gain G at one frequency within the filter bandpass, say 1000 Hz, you can use the SR770 spectrum analyzer to obtain the *shape* of G(f). See the SR770 file for instructions. You can then calibrate the SR770's relative G(f) to your spot measurement. We can help you with this.

JOHNSON NOISE FROM RESISTORS AT ROOM TEMPERATURE:

Disconnect the attenuator from the amplifier. Connect the resistor box to the amplifier and select a R value using the switch on the box. Measure the rms voltage of the amplified Johnson noise signal. Since the rms voltage fluctuates on the time scale of a fraction of second, it is difficult to obtain an accurate reading of the mean of the rms voltage. To obtain the latter, you need to integrate over time (see Agilent 34401A manual)

Measure the rms AC noise voltage of each of the resistors. The value of the resistance of each of the resistors is written on the amplifier box. If you wish to check the resistance of these resistors, you may by connecting an ohm meter to the resistor box connector and measuring the selected R. We encourage you to do this. ;)

Use Eq. (8) to calculate the Boltzmann constant k_B, taking into account the corrections mentioned below. Also, you should account for the amplifier noise, $\langle V_N^2 \rangle_{time}$, obtained from your data of the noise voltage measured at the amplifier output when the input is shorted. See equation 2: The noise generated in the amplifier's input transistor is a separate random noise that then adds in quadrature (square of uncorrelated noise voltages add). Be sure to refer these noise rms voltages to the input of your amplifier, by dividing the output rms noise voltage by the gain of the amplifier.

TEMPERATURE DEPENDENCE OF THE JOHNSON NOISE

A resistor in a stainless steel thermal probe tube is provided to explore the temperature dependence of Johnson noise. Connect this probe <u>directly</u> to the INPUT connector on the amplifier (additional cable will only add capacitance and microphonic noise). In fact, calculate what a little capacitance like 10pF might do to the shape of your noise spectrum! You may want to consider this in your data analysis.

Record the RMS voltage produced by this resistor at room temperature (about 300 K as measured with thermometer), water at the boiling point, and at liquid nitrogen (77 K). Also, use dry-ice for another temperature point.

Plot the rms noise voltage as a function of the temperature. Also, measure the resistance of the resistor at each of the temperatures (since the resistance of some resistors is a function of the temperature!).

If you find any discrepancy between the measurement and the theory, suggest what their source(s) might be.

Measurements of noise are very important to physics experiments, because the actual noise levels in the experiment can determine whether one can measure small signal levels in the experiment. Measurements of noise power spectra as described here are frequently performed to understand the sources of the noise in the experiment. If you understand the noise in your experiment, you can then work to reduce noise sources by, for example, choosing components with less noise, averaging longer to reduce the effects of noise on the signal, or working in frequency regions where the noise is lower. In fact, specialized frequency analyzers exist; these are instruments which can easily measure such noise spectra (see below for an example: the PicoScope.) Some modern digital sampling oscilloscopes have a useful FFT option, and can be used to explore a wide frequency spectrum. [However, most scopes are built for speed and only have 8 or 12 bit digitizers; you want more precision.] Not only are there other sources of noise, there are also other sources of interference which may introduce systematic errors. For example, the local FM station has a particularly strong signal in the lab (you should look for this and be sure it is not present at the low-level parts of your circuit by probing with a scope or high frequency spectrum analyzer. Even if your scope does not have a FFT option you can change the time base and sensitivity to see this sine wave if it is present.)

The PicoScope software will display the power spectral density of the noise as a function of frequency. Choose a log display with "auto" scaling.

This noise power spectrum measurement by a computer and fast Fourier Transform (PicoScope) is particularly useful for measuring the noise of the resistor in the separate probe as a function of temperature (room temperature (~300K), in crushed dry ice (195K), and in liquid nitrogen (77K). The noise from this resistor is particularly susceptible to microphonic noise. Microphonic noise is the noise voltage generated in electric wires due to their motion through capacitive effect or piezo-electric effect. Thus it can be generated from the probe being shaken, by people walking in the room

causing vibrations in the probe, etc. Measuring the noise power spectrum allows you to distinguish the Johnson noise (which is not frequency dependent) from microphonic noise, as well as electrical pickup from the AC power line frequency (60Hz) and noise from plasma discharges (lights!) that peak at specific frequencies such as multiples of 60Hz.

PICO-SCOPE SPECTRUM ANALYZER

For your experiment, you need to create your own folder to store the data files.

(1) **Measure thermal Johnson noise power spectra or $V^2(f)\Delta f$ using**
   The PicoScope and verify that thermal Johnson noise is indeed frequency-independent
   Question: How does the noise power spectrum $V^2(f)\Delta f$ compare to the square of the amplifier gain $G^2(f)$ that you measured ? Did you account for the RC filter on the input?

(2) **Determine the resistance and temperature dependence of the noise spectra and in turn calculate the Boltzmann constant k_B**
   Question: How does the noise power spectrum $V^2(f)\Delta f$ (shape and magnitude) vary with the resistance R ?
   Question: How does the noise power spectrum $V^2(f)\Delta f$ vary with temperature T ?
   Question: Can you think of a way to use the noise power spectrum and the measured gain curve $G^2(f)$ to calculate the Boltzmann constant k_B without the AC Voltmeter and the Integrating Voltmeter ?

(3) **Use Johnson noise spectra to determine the frequency response of the entire system.** Hint: this is a good way of measuring the combined filters of the input RC and the amplifier gain curve.

Notes

There are three units built specifically for this experiment. The first preamp box is combination amplifier and filter. A very low noise operational amplifier is used as the first stage of amplification for the Johnson noise. Three total amplification stages are used for a total voltage gain of about 10,000. Active filters are included with the amplifier to cut out the low and high frequencies, leaving a pass band covering approximately 100 Hz to 1.8 Khz.

The spectrum below shows the filter passband on the PicoScope spectrum analyzer. 10 log of the noise power is plotted vs frequency in KHz.The filter bandpass is not exactly rectangular, but you can correct for that. This plot shows the total noise spectrum output: a product of the amplifier gain curve and the input RC spectral roll-off due to the small parasitic capacitance.

**PicoScope spectrum of the Johnson noise in the integrating mode**

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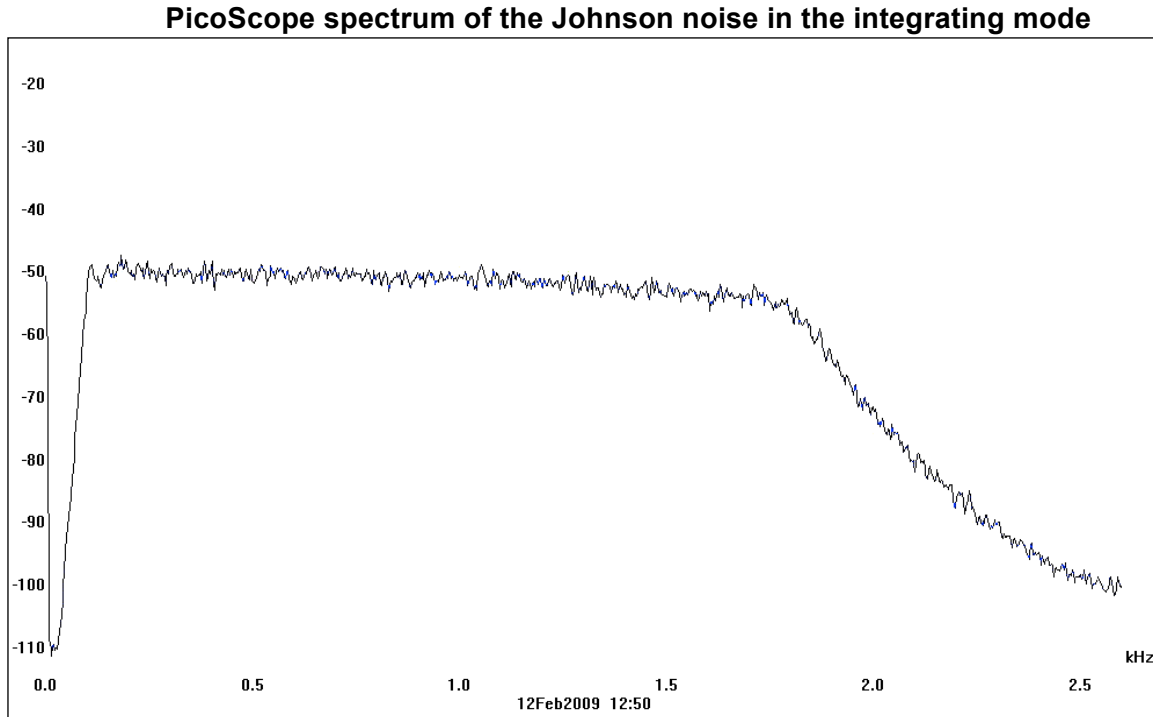
There is a resistor box that attaches to the amplifier input to allow testing for Johnson noise on several different values of resistors and a probe with a ~1 Megohm resistor (measure it!) that is used for testing of temperature effects on Johnson noise. When testing resistors for Johnson noise, the signal generator, frequency counter and attenuator are not used of course. When checking for amplifier/filter characteristics, the Generator, counter and attenuator are connected in place of the resistor box or probe.

Measuring the Gain Integral

Before taking data for Johnson noise, the characteristics of the Amplifier and Filter must be accurately determined. The setup for this is shown in the photo below.
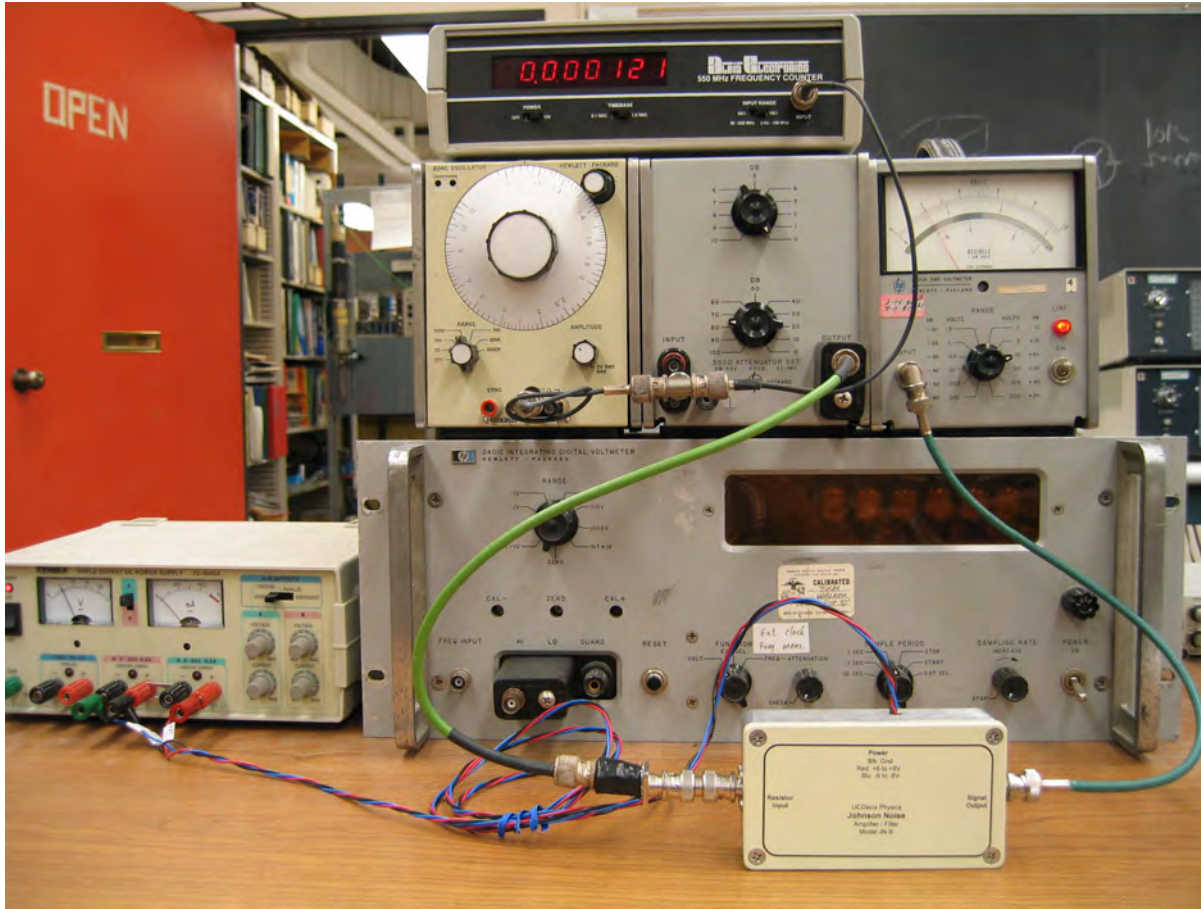
You can obtain the gain curve of the Amp/Filter by applying an input sine wave of known frequency and amplitude and then measuring the output voltage. Make sure the input amplitude is constant. Measuring the output at many frequencies across the band of interest and plotting the results on a frequency vs amplitude graph will provide results such as those on the next page. *Note that this is just the filter in the amplifier box; you must also consider the filter caused by the input capacitance in parallel with the resistor (see above total noise spectrum plot.)*
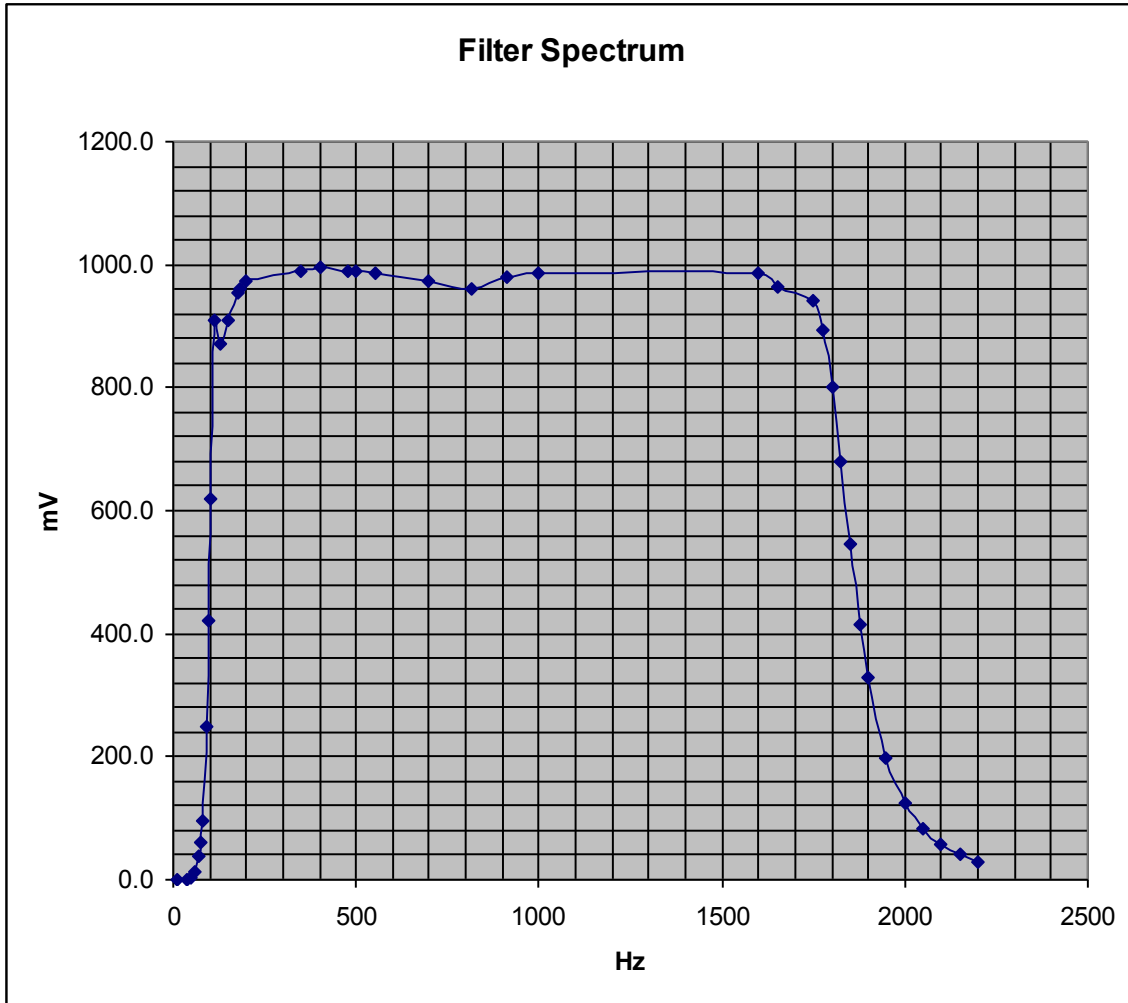
Filter Spectrum

If the filter response plot above simply had precisely vertical sides and a flat top, then the Johnson noise over the bandwidth would be the power spectral density times the bandwidth. Any filter response may be represented by an *effective* bandwidth which is the width (in Hz) of a rectangle fit to the response curve which has the same total area as the filter response curve.

What about internal noise generated in your amplifier? The full bandwidth internal noise can be measured by connecting a short circuit adapter (shunt) across the amplifier input and measuring the output voltage. Be sure to correct for this.

As mentioned before, there is an easier way to get the gain vs frequency. You can use the SR770 spectrum analyzer to give the shape of the Gain vs frequency. The spectrum below shows the filter passband on the SR770 spectrum analyzer. The gain is plotted vs frequency in KHz.The filter bandpass is not exactly rectangular, but you can correct for that. The SR770 generates a file with gain tabulated vs frequency 0-3 KHz. Read the SR770 file for instructions.

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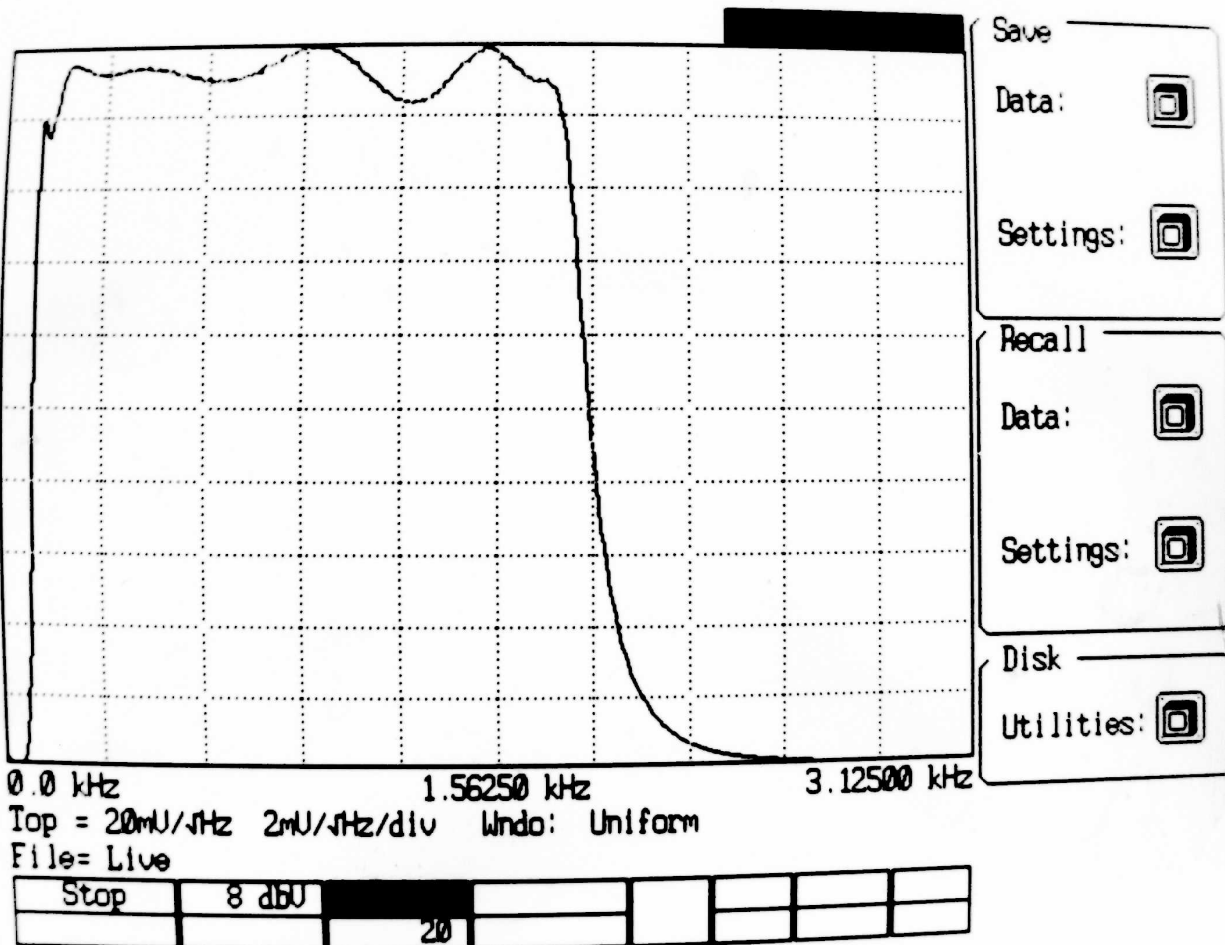
The amplifier filter response does not look like a nice rectangular box.
The effective bandwidth in the Johnson noise formula is simply the width Δf_{eff} of a rectangular passband which has total area $\int_0^\infty [G(f)]^2 df$.

i.e. $\Delta f_{eff} = \int_0^\infty [G(f)]^2 df \,/\, [G(f_0)]^2$ where f_0 is taken at the center of the passband.

The attenuator as well as the Picoscope use "dB power" (20 dB = factor of 100 in power, i.e. a factor of 10 in voltage). But do not trust their calibration. You must calibrate them yourself. The most accurate voltage standard you have is the Agilent digital rms voltmeter. Use it to calibrate the attenuator in the range you use it. When you do this, be sure to measure the input voltage to the attenuator while the oscillator is hooked into it (the oscillator's output voltage is loaded down by the attenuator's input resistance), and

then measure the output voltage from the attenuator in the attenuation range you will use it. You will have to adjust the oscillator output voltage to stay at the same input voltage to the attenuator for different attenuation settings. <u>You can also use the Agilent digital voltmeter to calibrate the picoscope dB power scale on its FFT spectrum.</u>

Most importantly, use the Agilent to make all voltage measurements directly, including rms noise voltage from the low noise bandpass amplifier. Use the Picoscope to visualize the spectrum and to make appropriate corrections to the rms voltage measured by the Agilent. You will find that you need to do this correction only in two cases: very high resistance (because of the change in gain profile which you see on the Picoscope spectrum), and at liquid nitrogen temperature (because of microphonic bursts of power in acoustic resonances which show nicely on your Picoscope spectrum).

You should plot mean square voltage vs resistance at constant temperature, and mean square voltage vs temperature at constant resistance. Look at the intercepts. You will be able to separate out the noise generated by the amplifier and the Johnson noise from the resistor. From these data you will be able to derive Boltzmann's constant.

REFERENCES

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- Kittel, *Thermal Physics*, pp. 98-102
- Kittel, *Elementary Statistical Physics*, pp. 141-149.
- Nyquist, *Phys. Rev.* 32, 110 (1928)