

## 1 Introduction

A Transmission line is a pair of conductors which have a cross which remains constant with distance. For example, a coaxial cable transmission line has a cross section of a central rod and an outer concentric cylinder.

Similarly a twisted pair transmission line has two conducting rods or wires which slowly wind around each other. A cross section made at any distance along the line is the same as a cross section made at any other point on the line.

We want to understand the voltage - Current relationships of transmission lines.

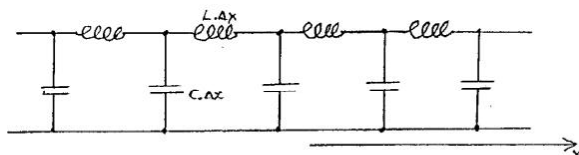
## 2 Equations for a “lossless” Transmission Line

A transmission line has a distributed inductance on each line and a distributed capacitance between the two conductors. We will consider the line to have zero series resistance and the insulator to have infinite resistance (a zero conductance or perfect insulator). We will consider a “Lossy” line later in section 12 on page 25.

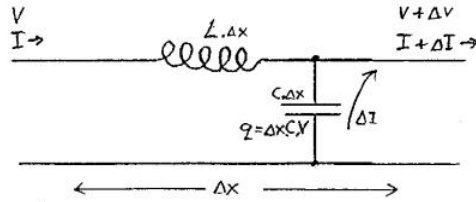
Define  $L$  to be the inductance/unit length and  $C$  to be the capacitance/unit length.

Consider a transmission line to be a pair of conductors divided into a number of cells with each cell having a small inductance in one line and having small capacitance to the other line.

In the limit of these cells being very small, they can represent a distributed inductance with distributed capacitance to the other conductor.



Consider one such cell corresponding to the components between position  $x$  and position  $x + \Delta x$  along the transmission line.



The small series inductance is  $L \cdot \Delta x$  and the small parallel capacitance is  $C \cdot \Delta x$ .

Define the voltage and current to the right on the left side to be  $V$  and  $I$ . Define the voltage and current to the right on the right side to be  $V + \Delta V$  and  $I + \Delta I$ .

We now can get two equations.

1. The current increment  $\Delta I$  between the left and right ends of the cell is discharging the capacitance in the cell.

The charge on the cell's capacitance = capacitance x voltage =  $C \cdot \Delta x \cdot V$  and so the current leaving the capacitance to provide  $\Delta I$  must be;

$$\Delta I = -\frac{\partial}{\partial t}(\text{Charge}) = -\frac{\partial}{\partial t}(C \cdot \Delta x \cdot V)$$

The minus sign is due to the current leaving the capacitor.

$$\Delta I = -C \cdot \Delta x \cdot \frac{\partial V}{\partial t}$$

$$\frac{\Delta I}{\Delta x} = -C \cdot \frac{\partial V}{\partial t}$$

Note the minus sign.

2. The voltage increment  $\Delta V$  between the left and right ends of the cell is due to the changing current through the cell's inductance. (Lenz's Law)

$$\Delta V = -\text{Inductance} \cdot \frac{\partial I}{\partial t} = -\Delta x \cdot L \cdot \frac{\partial I}{\partial t}$$

$$\frac{\Delta V}{\Delta x} = -L \cdot \frac{\partial I}{\partial t}$$

Now take the limit of the cell being made very small so that the inductance and capacitance are uniformly distributed. The two equations then become

$$\frac{\partial I}{\partial x} = -C \cdot \frac{\partial V}{\partial t} \quad \text{Equation 1.}$$

$$\frac{\partial V}{\partial x} = -L \cdot \frac{\partial I}{\partial t} \quad \text{Equation 2.}$$

Remember that  $L$  and  $C$  are the inductance/unit length measured, in Henries/meter and are the capacitance/unit length measured in Farads/meter.

Differentiate equation 2 with respect to the distance  $x$ .

$$\frac{\partial}{\partial x} \left( \frac{\partial V}{\partial x} \right) = -L \cdot \frac{\partial}{\partial x} \left( \frac{\partial I}{\partial t} \right)$$

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial x} \left( \frac{\partial I}{\partial t} \right)$$

x and t are independent variables and so the order of the partials can be changed.

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial t} \left( \frac{\partial I}{\partial x} \right)$$

Now substitute for  $\frac{\partial I}{\partial x}$  from equation 1 above

$$\frac{\partial^2 V}{\partial x^2} = -L \cdot \frac{\partial}{\partial t} \left( -C \cdot \frac{\partial V}{\partial t} \right)$$

$$\frac{\partial^2 V}{\partial x^2} = LC \cdot \frac{\partial^2 V}{\partial t^2} \quad \text{Equation 3}$$

This is usually called the Transmission Line Differential Equation.

Notes

- L and C are NOT just the inductance and the capacitance. They are both measured per unit length.
- The Transmission Line Differential Equation 3 above does NOT have a minus sign.

The Transmission Line Differential Equation 3 above is a normal 1 dimensional wave equation and is very similar to other wave equations in physics. From experience with such wave equations, we can try the normal solution of the form

$$V = V(s)$$

where s is a new variable  $s = x + ut$ . Substituting this into the two sides of the Transmission Line Differential Equation 3 above we get the two sides being

$$\frac{\partial^2 V}{\partial x^2} \text{ and } \frac{1}{u^2} \cdot \frac{\partial^2 V}{\partial t^2}$$

Thus the form  $V(x + ut)$  can satisfy the Transmission Line Differential Equation 3 if and only if

$$\frac{1}{u^2} = LC \quad \text{Equation 4.}$$

Both roots of this satisfy the Equation 3.

$$u = \pm \frac{1}{\sqrt{LC}}$$

The two roots give slightly different solutions and so, since the equation 3 is linear, any linear combination of the two solutions is a valid solution. Define u as the positive root

$$u = + \frac{1}{\sqrt{LC}} \quad \text{Equation 5.}$$

### 3 The Voltage Solution

Thus, the general solution for the voltage is the linear combination.

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

Where  $f()$  and  $g()$  are arbitrary single valued functions which can be very different.

1.  $f(x - ut)$  describes a wave propagating with no change in shape towards  $x = +\infty$ .
2.  $g(x + ut)$  describes a wave propagating with no change in shape towards  $x = -\infty$ .

### 4 The Current Solution

Consider one of the waves such as the “forward wave” propagating towards  $x = +\infty$ .

$$V = V(x - ut)$$

From this we can show, by differentiating, that:

$$\begin{aligned} -u \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial t} \\ \frac{\partial V}{\partial x} &= -\frac{1}{u} \cdot \frac{\partial V}{\partial t} \end{aligned} \quad \text{Equation 7.}$$

Also from equation 2. above

$$\frac{\partial V}{\partial x} = -L \cdot \frac{\partial I}{\partial t} \quad \text{Equation 2.}$$

Equation 2 and equation 7 will have a common solution only if the two right hand sides are the same

$$\frac{1}{u} \cdot \frac{\partial V}{\partial t} = L \cdot \frac{\partial I}{\partial t}$$

$$V = uL \cdot I$$

This can be rewritten using  $u = \frac{1}{\sqrt{LC}}$  from equation 5.

$$V = \sqrt{\frac{L}{C}} \cdot I$$

and

$$\frac{V}{I} = \sqrt{\frac{L}{C}}$$

and the current  $I$  of the forward wave is

$$I = V / \sqrt{\frac{L}{C}}$$

and, similarly for the backward wave

$$V = -\sqrt{\frac{L}{C}} \cdot I$$

$$\frac{V}{I} = -\sqrt{\frac{L}{C}}$$

and the current  $I$  of the backward wave is

$$I = -V/\sqrt{\frac{L}{C}}$$

Thus the general solution for both waves for the current  $I$  is

$$I = (f(x - ut) - g(x + ut))/\sqrt{\frac{L}{C}} \quad \text{Equation 7}$$

which can be compared with the earlier equation for the voltage

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

## 5 The “Characteristic Impedance $Z_0$ ”

Define the “Characteristic Impedance  $Z_0$ ” as the magnitude of the instantaneous ratio for either the forward wave or backward wave. For the forward wave:

$$Z_0 = \left| \frac{\text{Voltage}}{\text{Current}} \right| = \left| \frac{V}{I} \right| = \left| \sqrt{\frac{L}{C}} \right|$$

For the backward wave:

$$Z_0 = \left| \frac{\text{Voltage}}{\text{Current}} \right| = \left| \frac{V}{I} \right| = \left| \sqrt{\frac{L}{C}} \right|$$

With this definition of  $Z_0$  the voltage and current equations can be written:

$$V = f(x - ut) + g(x + ut) \quad \text{Equation 6.}$$

$$I = \frac{f(x-ut)}{Z_0} - \frac{g(x+ut)}{Z_0} \quad \text{Equation 8.}$$

## 6 Speed $u$ of Signals

The Inductance per unit length  $L$  and Capacitance per unit length  $C$  can be calculated from Electromagnetic Theory. The formulae depend upon the cross sectional shape of the conductors.

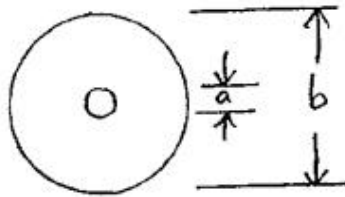
## 7 Impedances of Actual Cables

### • Coaxial Cable

EM theory says that a Coaxial Cable with inner rod having diameter  $a$  and outer tube having diameter  $b$  has

$$C = \frac{2\pi\epsilon}{\ln(\frac{b}{a})} \text{ and}$$

$$L = \frac{\mu}{2\pi} \ln(\frac{b}{a}).$$



From these,  $u$  and  $Z_0$  can be

obtained

$$u = \frac{1}{\sqrt{\mu\epsilon}} \text{ and}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln(\frac{b}{a}) \approx \frac{60}{\epsilon_r} \ln(\frac{b}{a})$$

For a vacuum, of course,

$$u_{vac} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \text{Velocity of light and}$$

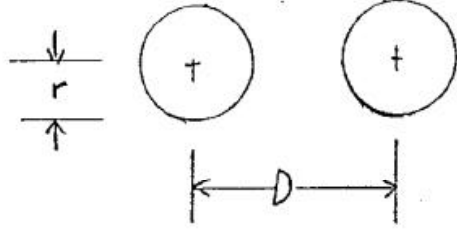
for polyethylene,  $\epsilon_r$  is about 2.3, and the speed is almost halved.

### • Parallel Cylinders

EM theory says that a pair of parallel conducting cylinders, rods or wires, with rod radius  $r$  and center to center separation  $D$  have

$$C = \frac{\pi\epsilon}{\ln(\frac{D}{r})}$$

$$L = \frac{\mu}{\pi} \ln(\frac{D}{r})$$



From these,  $u$  and  $Z_0$  can be obtained  $u = \frac{1}{\sqrt{\mu\epsilon}}$  and  $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln\left(\frac{D}{r}\right) \approx \frac{240}{\epsilon_r} \ln\left(\frac{D}{r}\right)$

Note that

1. in both cases the speed  $u$  is the same and depends only upon the medium. In fact, of course, for all possible cross sectional shapes, the speed is the same  $u = \frac{1}{\sqrt{\mu\epsilon}}$  and if the medium is a vacuum,  $u = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$  the speed of the voltage/current signal on the cable is the speed of light.
2. in both cases, the speed and the Characteristic Impedance depend upon logarithms of the ratio of two distances in the cross.
  - thus a big transmission line can have the same impedance as a small transmission line if one is scaled in proportion from the other.
  - For most lines it is not practical to vary the ratios  $\frac{b}{a}$  and  $\frac{D}{r}$  much more than about 2.0/1 up to 10/1. Since the  $\ln\left(\frac{2}{1}\right) \approx 0.69$  and  $\ln\left(\frac{10}{1}\right) \approx 2.3$  the range of impedances is normally within a moderate range (about 20 ohm to 200 ohm).

Some typical values can be found for  $Z_0 = 50 \text{ ohm}$  cable with a polyethylene dielectric and speed  $u = \text{half of light} = 1.5 \times 10^8 \text{ m/s}$ . Use the speed equation from equation 5 of section 2 on page 4 and the characteristic impedance equation from section 5 on page 6.

$$u = \frac{1}{\sqrt{LC}} \text{ and } Z_0 = \sqrt{\frac{L}{C}}$$

By multiplying and dividing these equations, we can get  $L$  and  $C$

$$L = \frac{Z_0}{u} \text{ and } C = \frac{1}{Z_0 u}$$

For the values assumed for  $Z_0$  and  $u$

$$\begin{aligned} L &= \frac{50 \text{ ohm}}{1.5 \times 10^8 \text{ m/s}} = 3.333 \times 10^{-7} \text{ Henry/meter} \\ &= 333 \times 10^{-9} \text{ Henry/meter} = 333 \text{ nanoHenry/meter} \end{aligned}$$

$$C = \frac{1}{50 \text{ ohm} \times 1.5 \times 10^8 \text{ m/s}} = \frac{1}{75 \times 10^8} \text{ Farad/m}$$
$$= 1.333 \times 10^{10} \text{ Farad/m} = 133.3 \text{ pF/m}$$

Thus a foot of RG58 cable with  $Z=50$  ohm and  $u$ =half of light has a capacitance of  $\approx 0.305\text{m/ft} \times 1.333\text{pF/m} = 40 \text{ pF}$ .

There are many types of RG58 50 ohm coax, with a range of loss, and the type of cable you use in PHY 122 lab will have slightly different capacitance and inductance per unit length than the values used in the exercise above.